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CLASSIFICATION OF ALL IRREDUCIBLE UNITARY REPRESENTATIONS OF THE STABILIZER OF THE HORICYCLES OF A TREE

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ABSTRACT

In this paper we describe all irreducible unitary representations of the stabilizer of the horicycles associated with an end of a homogeneous tree.

Let X be a homogeneous tree of finite order $r+1 \geqq 3$ and let ω be an element of the tree boundary Ω. Let $\{H_n\}$, $n \in \mathbf{Z}$ be the partition of X into the horicycles associated with ω (for definitions and more details we refer the reader to [6]). Let B_ω be the stabilizer in Aut(X), the group of all isometries of X, of the partition $\{H_n\}$; B_ω is closed in Aut(X). For a fixed point $x_0 \in X$, let $[x_0, \omega) = \{x_0, x_1, \ldots, x_n, \ldots\}$ be the geodesic connecting x_0 to ω. It is easy to see that B_ω is the group of isometries b such that $b(x_n) = x_n$ for n sufficiently large. Let K_v be the stabilizer in Aut(X) of a vertex v of X and $B_n = B_\omega \cap K_{x_n}$; B_n is compact open in B_ω, $B_n \subset B_{n+1}$ and $B_\omega = \bigcup_{n=0}^{+\infty} B_n$. In particular B_ω is a unimodular amenable group. We will fix from now on another end of the tree $\omega' \neq \omega$ so as to identify a fixed infinite geodesic

$$(\omega', \omega) = \{\ldots, x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots\}.$$

Then, for every $n \in \mathbf{Z}$, $B_n = B_\omega \cap K_{x_n}$, $B_n \subset B_{n+1}$ and $B_\omega = \bigcup_{n=-\infty}^{+\infty} B_n$.

In this paper we will classify the irreducible unitary representations of B_ω using the method of Ol'shianskii [4]. We summarize, briefly, this method.

Let Δ be a finite complete subtree (i.e. every vertex of Δ is of homogeneity $r+1$ or 1 in Δ); let $K(\Delta) = \{g \in \mathrm{Aut}(X); g(x) = x \text{ for every } x \in \Delta\}$. If G is a

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closed subgroup of Aut(X), then the sets $G(\Delta) = G \cap K(\Delta)$ form a basis of open neighborhoods of unity of G.

Let π be a unitary representation of G; we define

$$P_\pi(\Delta) = (1/\text{mis}(G(\Delta)) \int_{G(\Delta)} \pi(g)dg$$

where dg is the Haar measure of G and mis($G(\Delta)$) is the measure of $G(\Delta)$. $P_\pi(\Delta)$ is a projection of H, the Hilbert space of π. More precisely $P_\pi(\Delta)$ is the projection on the closed subspace $H(\Delta)$ of H consisting of $G(\Delta)$-invariant vectors, that is of vectors $v \in H$ such that $\pi(g)v = v$ for every $g \in G(\Delta)$. Since the sets $G(\Delta)$ are a basis of open neighborhoods of unity of G, for $v \in H$, $v \neq 0$ there exists Δ such that $\int_{G(\Delta)} (\pi(g)v, v)dg \neq 0$, that is $P_\pi(\Delta)v \neq 0$.

Let $l_\pi = \min\{\text{card}(\Delta) : P_\pi(\Delta) \neq 0, \Delta \text{ is a complete finite subtree}\}$.

Definition. A unitary irreducible representation π of G is called spherical if $l_\pi = 1$, special if $l_\pi = 2$ and cuspidal if $l_\pi > 2$.

A finite complete subtree Δ is called minimal for π if $P_\pi(\Delta) \neq 0$ and card(Δ) $= l_\pi$. If Δ is minimal, then $g(\Delta)$ is minimal for every $g \in G$. In the case of B_ω, every π is either spherical or cuspidal; in fact, if $[a, b]$ is an edge and $\{a, b, s_1, s_2, \ldots\}$ is the geodesic from a to ω, then $B_\omega \cap K(\{a\}) \subsetneq B_\omega \cap K([a, b])$. Therefore $P_\pi([a, b]) \neq 0$ implies that $P_\pi(\{a\}) \neq 0$ and so $l_\pi = 1$. Thus $l_\pi \neq 2$ for every unitary representation of B_ω. Moreover if $\{x\}$ is a minimal subtree for the spherical representation π, then there exists $b \in B_\omega$ such that $b(x) = x_n$ for some n (recall that $\{x_n\}$, $n \in \mathbf{Z}$ is the geodesic from ω' to ω and B_ω acts transitively on $\Omega \setminus \{\omega\}$); in particular $\{x_n\}$ is a minimal subtree for π. This means that we can describe the unitary irreducible spherical representations of B_ω in terms of the minimal subtrees $\{x_n\}$, $n \in \mathbf{Z}$. Since $B_n \subset B_{n+1}$, if $\{x_n\}$ is a minimal subtree for π, then also $\{x_m\}$ is a minimal subtree for every $m < n$.

The spherical irreducible representations of B_ω

We recall that a pair (G, K), where G is a locally compact group and K is a compact subgroup of G, is called a Gelfand pair if the algebra $L^1(G/K)$ consisting of bi $-$ K-invariant L^1-functions is abelian. A representation π of G is called spherical (in the sense of Gelfand pairs) if π has a nontrivial K-invariant vector. For every $n \in \mathbf{Z}$, it is easy to see that (B_ω, B_n) is a Gelfand pair. In fact, if $b \in B_k \setminus B_{k-1}$ and $n < k$, then $d(b(x_n), x_k) = d(b^{-1}(x_n), x_k) =$

$d(x_n, x_k)$ and $b(x_n) \neq x_n$, $b^{-1}(x_n) \neq x_n$. Therefore there exists h in B_n such that $h(b(x_n)) = b^{-1}(x_n)$; this implies that $bhb \in B_n$ and so $b^{-1} \in B_n b B_n$ for every $n < k$. Since $b \in B_k \setminus B_{k-1}$ we have that $b^{-1} \in B_n b B_n$ for every $b \in B_\omega$ and for every $n \in \mathbf{Z}$. By [1, IV-2, th. 1], (B_ω, B_n) is a Gelfand pair for every $n \in \mathbf{Z}$. Therefore the unitary irreducible spherical representations of B_ω correspond to the spherical representations of the Gelfand pairs (B_ω, B_n) for $n \in \mathbf{Z}$, and so to the positive definite spherical functions of (B_ω, B_n) [1, IV-5, corollary of th. 9].

PROPOSITION 1. *The nontrivial spherical unitary irreducible representations of B_ω are L^1-representations; in fact there exists a dense subspace of coefficients with compact support.*

The spherical functions of B_ω are the following: ϕ_∞ the function identically one and $\phi_n = \chi_{B_n} + (1-r)^{-1}\chi_{(B_{n+1}\setminus B_n)}$ for every $n \in \mathbf{Z}$. (χ_E is the characteristic function of the set E.)

The unitary irreducible spherical representations of B_ω are the following: π_∞ the trivial one-dimensional representation, and π_n the subrepresentations of the left regular representation on $L^2(B_\omega)$ generated by the functions $\phi_n \in L^2(B_\omega)$ for $n \in \mathbf{Z}$.

For $n \in \mathbf{Z}$, the formal dimension of π_n is equal to $(r-1)/(\mathrm{mis}(B_0))r^{n+1}$

PROOF. Since every spherical function ϕ with compact support is positive definite (in fact by [1, IV-3, definition], $\phi * \phi^* = \|\phi\|_2^2 \phi$), to prove the Proposition it remains only to prove that ϕ_n are all the spherical nontrivial functions of B_ω. More precisely, it is enough to prove that for every fixed $n_0 \in \mathbf{Z}$, the spherical nontrivial functions of the Gelfand pair (B_ω, B_{n_0}) are the following:

$$\phi_{n_0+k} = \chi_{B_{n_0+k}} + (1-r)^{-1}\chi_{(B_{n_0+k+1} \setminus B_{n_0+k})} \qquad \text{for every } k \geqq 0.$$

To prove this, we observe that for every $n \in \mathbf{Z}$, B_{n-1} is a subgroup of index r of B_n, so $\mathrm{mis}(B_n) = r\,\mathrm{mis}(B_{n-1})$. We choose the Haar measure of B_ω in such a way that B_0 has measure equal to one. Therefore $\mathrm{mis}(B_n) = r^n$. The set $B_n \setminus B_{n-1}$ is compact open and $\mathrm{mis}(B_n \setminus B_{n-1}) = (r-1)\mathrm{mis}(B_{n-1}) = (r-1)r^{n-1}$. Let $\lambda_n = \mathrm{mis}(B_n \setminus B_{n-1})$, and $\mu_n = (\lambda_n)^{-1}\chi_{B_n \setminus B_{n-1}}$. It is easy to see that the bi $-B_{n_0}$-invariant functions of B_ω are, exactly, the linear combinations of the functions μ_n, $n > n_0$ and $\chi_{B_{n_0}}$. In particular every bi $-B_{n_0}$-invariant function is continuous.

It follows directly that $\mu_n * \mu_m = \mu_m$ for $n < m$, while an easy computation shows that

$$\mu_n * \mu_n = (\lambda_n)^{-1}\chi_{B_{n-1}} + [(r-2)/(r-1)]\mu_n.$$

Let $\langle f, g\rangle = \int_{B_\omega} f(x)g(x)dx$. We recall that a bi $- B_{n_0}$-invariant function ϕ not identically zero is spherical iff $\langle \phi, f*g\rangle = \langle \phi, f\rangle\langle \phi, g\rangle$ for every f, g bi $- B_{n_0}$-invariant functions with compact support [1, IV-3, th. 6]. Let $\phi(n)$ be the value on $B_n \setminus B_{n-1}$ for $n > n_0$ and $\phi(n_0)$ the value on B_{n_0}; then $\mu_n * \mu_m = \mu_m$ implies that $\phi(n)\phi(m) = \phi(m)$ for every $n < m$. This implies that if $\phi(n) = 0$ then $\phi(m) = 0$ for every $m \geqq n$; moreover, if $\phi(m) \neq 0$ then $\phi(n) = 1$ for every $n < m$. This means that if ϕ is spherical not identically one then, for some $k \geqq 0$, we have

$$\phi = \chi_{B_{n_0+k}} + \alpha\chi_{(B_{n_0+k+1}\setminus B_{n_0+k})}.$$

The fact that $\phi * \mu_n = c_n\phi$ for every $n > n_0$ implies that $\alpha = (1-r)^{-1}$, as is easily seen. On the other hand, the functions ϕ_{n_0+k}, for every $k \geqq 0$, are spherical functions of the pair (B_ω, B_{n_0}) because an easy computation shows that ϕ_{n_0+k} satisfies the equations $\phi * \mu_n = c_n\phi$ for $n > n_0$ and $\phi * \chi_{B_{n_0}} = c_0\phi$. We recall that the formal dimension of π_n is equal to $\| \phi_n \|_2^{-2}$. The Proposition follows.

The cuspidal irreducible representations of B_ω

In [4] Ol'shianskii gives the classification of all irreducible unitary represetations of Aut(X), where X is a homogeneous or semihomogeneous tree. We can apply the method of Ol'shianskii [4, th. 2.2] to describe the cuspidal unitary irreducible representations of B_ω. In fact we can characterize the irreducible cuspidal representations for a larger class of closed subgroups of Aut(X) which have been introduced by Tits in [6].

The crucial step of the method of Ol'shianskii [4, lemma 2.4] is the following property (*) for a closed subgroup G of Aut(X):

(*) *For every finite subtree* Δ *of* X *with* card $\Delta \geqq 2$, *for every connected component* E *of* $X \setminus \Delta$, *and for every* $g \in G(\Delta)$ *there exists* $k \in G(\Delta)$ *such that* $k(x) = g(x)$ *for* $x \in E$ *and* $k(x) = x$ *for* $x \notin E$.

If a closed subgroup G of Aut(X) has property (*), then Lemma 2.4 of [4] is true with the same proof and we can prove, *mutatis mutandis*, theorem 1 of [4, section 2 Cuspidal representations].

In fact, let Δ be a complete finite subtree of X with diam $\Delta \geqq 2$; let $M(\Delta)$ be the subspace of $L^2(G)$ consisting of $G(\Delta)$ right-invariant functions such that the right averaging over $G(\Delta_0)$ is zero for every complete subtree $\Delta_0 \subsetneq \Delta$ (i.e. $P_\rho(\Delta_0)f = 0$ for every $\Delta_0 \subsetneq \Delta$, where ρ is the right regular representation of G). $M(\Delta)$ is a closed left-invariant subspace of $L^2(G)$ ($M(\Delta)$ is the closure in $L^2(G)$ of the space $H(\Delta)$ of [4, 2.4]). Let $\Delta_1, \Delta_2, \ldots, \Delta_j$ all be complete maximal subtrees of Δ different from Δ. Let $\tilde{G}(\Delta) = \{g \in G : g\Delta = \Delta\}$. Obviously $G(\Delta) \subset G(\Delta_i) \subset \tilde{G}(\Delta)$ for every $i = 1, 2, \ldots, j$; $G(\Delta)$ is a normal subgroup of $\tilde{G}(\Delta)$ and $\tilde{G}(\Delta)$ is compact. The inner automorphisms of $\tilde{G}(\Delta)$ permute the subgroups $G(\Delta_i)$. Let $(\tilde{G}(\Delta))_0^\wedge$ be the set consisting of all classes of unitary irreducible representations π of $\tilde{G}(\Delta)$ such that π is trivial on $G(\Delta)$ and it has no nonzero $G(\Delta_i)$-invariant vectors for $i = 1, 2, \ldots, j$. In other words, $(\tilde{G}(\Delta))_0^\wedge$ is the set of classes of irreducible representations of the finite group $\tilde{G}(\Delta)/G(\Delta)$ without nonzero $G(\Delta_i)/G(\Delta)$-invariant vectors for every $i = 1, 2, \ldots, j$ [4, 2.1].

For a closed subgroup G of $\mathrm{Aut}(X)$ with property (∗), we have, as in [4, lemma 2.4, 2.2 theorem 1], that the following are equivalent:

(a) $M(\Delta) \neq 0$,

(b) $(\tilde{G}(\Delta))_0^\wedge \neq \varnothing$,

(c) there exist unitary cuspidal irreducible representations of G with minimal tree Δ.

In particular, if $\operatorname{diam} \Delta \geqq 2$ and $(\tilde{G}(\Delta))_0^\wedge \neq \varnothing$ then $G(\Delta) \neq G(\Delta_i)$ for every $i = 1, 2, \ldots, j$, because in this case the groups $G(\Delta_i)/G(\Delta)$ are nontrivial. We shall see that this condition is also equivalent to the fact that $(\tilde{G}(\Delta))_0^\wedge \neq \varnothing$ for $\operatorname{diam} \Delta > 2$ and G with property (∗).

If there exist cuspidal irreducible representations of G with minimal tree Δ, then $M(\Delta) \neq 0$ and these representations are, exactly, the inequivalent irreducible subrepresentations of $\lambda(\Delta)$ where $\lambda(\Delta)$ is the subrepresentation of the left regular representation of G relating to the left invariant subspace $M(\Delta)$. As in [4, theorem 1], these representations are the induced representations of $(\tilde{G}(\Delta))_0^\wedge$.

We summarize these facts in the following Proposition 2. As observed, the proof is similar to the proof of [4, theorem 1 section 2 Cuspidal representations] and so we omit it.

Proposition 2. *Let G be a closed subgroup of* $\mathrm{Aut}(X)$ *which has property* (∗); *let Δ be a complete finite subtree of X with* $\operatorname{diam} \Delta \geqq 2$. *Then the classes of unitary irreducible cuspidal representations of G with minimal tree Δ correspond, bijectively, to the induced representations of $(\tilde{G}(\Delta))_0^\wedge$, that is the inequivalent cuspidal irreducible representations of G with minimal tree Δ are* *the representations of the set* $\{\mathrm{ind}\,\sigma : \sigma \in (\tilde{G}(\Delta))_0^\wedge\}$. *Every cuspidal irreducible representation of G has a dense subspace of coefficients with compact support.*

If, in addition, G is unimodular then the cuspidal irreducible representations are L^1*-representations and the formal dimension of* ind σ *is equal to* $\dim \sigma / mis(\tilde{G}(\Delta))$.

If π is a cuspidal representation relating to Δ, then [4, 2.4, Lemma] implies that Δ' is a minimal tree for π if and only if $\Delta' = g\Delta$ for some $g \in G$. In particular, if π_1 is unitarily equivalent to π_2 then $\Delta_1 = g\Delta_2$ for some $g \in G$.

We provide now a necessary and sufficient condition for the existence of irreducible cuspidal representations of G.

THEOREM 3. *Let G be a closed subgroup of* Aut(X) *which has property* (*), *let* Δ *be a complete subtree of X with* diam $\Delta > 2$. *Then there exist irreducible cuspidal representations of G with minimal tree* Δ *if and only if* $G(\Delta) \neq G(\Delta_i)$ *for every* $i = 1, 2, \ldots, j$, *where* $\Delta_1, \Delta_2, \ldots, \Delta_j$ *are the complete maximal subtrees of* Δ, *different from* Δ.

PROOF. As observed, the condition $G(\Delta) \neq G(\Delta_i)$ for every i is a necessary condition for every Δ with diam $\Delta \geqq 2$. Hence, it is enough to prove that $G(\Delta) \neq G(\Delta_i)$ for every $i = 1, 2, \ldots, j$ and diam $\Delta > 2$ imply that $(\tilde{G}(\Delta))_0^\wedge \neq \emptyset$. To prove this, let Δ^0 be the subtree of Δ consisting of all vertices of Δ of homogeneity $r + 1$ in Δ. Since diam $\Delta > 2$, Δ^0 is a finite subtree of Δ with card $\Delta^0 \geqq 2$. Therefore, property (*) implies that $G(\Delta^0)/G(\Delta)$ decomposes into the direct product of its subgroups $G(\Delta_i)/G(\Delta)$. The inner automorphisms of $\tilde{G}(\Delta)/G(\Delta)$ permute the nontrivial subgroups $G(\Delta_i)/G(\Delta)$ and so Theorem 3 is a consequence of the following Lemma 4.

LEMMA 4. *Let G be a finite group and H a subgroup of G. Let* $H_1, H_2, \ldots, H_j$ *be nontrivial subgroups of G such that*:

(i) *H is the direct product of* $H_1, H_2, \ldots, H_j$;

(ii) *the inner automorphisms of G permute the subgroups* $H_1, H_2, \ldots, H_j$. *Then there exists a unitary irreducible representation of G which has no nonzero* H_i *invariant vectors for every* $i = 1, 2, \ldots, j$.

PROOF. First, we observe that such a representation exists iff there exists a function f not identically zero on G such that $\Sigma_{h \in H_i} f(ght) = 0$ for every $g, t \in G$ and for every $i = 1, 2, \ldots, j$. Indeed let M be the set of functions of this type; obviously M is a left-invariant finite-dimensional space; every irreducible subrepresentation of M has no H_i-invariant vectors for every $i = 1, 2, \ldots, j$.

Conversely, every left-invariant space N of functions on G without left H_i-invariant nontrivial functions for every $i = 1, 2, \ldots, j$ is contained in M because $P_\lambda(H_i)f(t) = (1/|H_i|)\Sigma_h f(ht) \in N$ and so $P_\lambda(H_i)f = 0$ for every $f \in N$, in particular for every $\lambda(g)f$, $f \in N$, $g \in G$. This proves the claim. Hence, to prove the Lemma it is enough to prove that there exists a function $F \neq 0$ on G such that $\Sigma_{h \in H_i} F(gh) = 0$ for $i = 1, 2, \ldots, j$ and $g \in G$ because the assumptions imply that every set $gH_i t$ is a right coset of H_k for some k. Moreover, it is enough to prove the Lemma in the special case $G = H$, in fact the function $F = f\chi_H$, where $f \neq 0$ on H is the function relating to H, satisfies the requirements.

We prove now that there exists a function $f \neq 0$ on $H = H_1 \times H_2 \times \cdots \times H_j$ such that $\Sigma_{h \in H_i} f(gh) = 0$ for every $i = 1, 2, \ldots, j$, and for every $g \in H$.

Let $E_i = \{x_i, y_i\} \subset H_i$, $x_i \neq y_i$ for every $i = 1, 2, \ldots, j$; let $E = E_1 \times \cdots \times E_j \subset H_1 \times H_2 \times \cdots \times H_j$. For $v \in E$, let $N(v)$ be the number of x_i occurring in the co-ordinates of $v \in E$. We define $f(v) = (-1)^{N(v)}$ for $v \in E$ and $f(v) = 0$ elsewhere. Therefore $f \neq 0$ on H; if $g = (\sigma_1, \sigma_2, \ldots, \sigma_j)$ then the coset $gH_i = \{(\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, h, \sigma_{i+1}, \ldots, \sigma_j) : h \in H_i\}$. Therefore if there exists k, $1 \leqq k \leqq j$, $k \neq i$ such that $\sigma_k \notin E_k$, then $gH_i \cap E = \emptyset$ and f is identically zero on gH_i. If $\sigma_k \in E_k$ for every $k \neq i$, then

$$gH_i \cap E = \{(\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, x_i, \sigma_{i+1}, \ldots, \sigma_j), (\sigma_1, \ldots, \sigma_{i-1}, y_i, \sigma_{i+1}, \ldots, \sigma_j)\}.$$

Since $N(\sigma_1, \ldots, x_i, \ldots, \sigma_j)) = N((\sigma_1, \ldots, y_i, \ldots, \sigma_j)) + 1$ we have that $\Sigma_{h \in H_i} f(gh) = 0$ and the Lemma is proved.

We provide now examples of groups with property (*). Let $\mathscr{L}$ be a partition of the tree (or of the tree boundary Ω). We define $\mathrm{Aut}_{\mathscr{L}}(X)$ as the group of all isometries g of X such that for every $x \in X$, x and $g(x)$ belong to the same set of the partition $\mathscr{L}$ (such that for every $\omega \in \Omega$, ω and $g\omega$ belong to the same set of the partition $\mathscr{L}$ of Ω). It is easy to see that $\mathrm{Aut}_{\mathscr{L}}$ is a closed subgroup of $\mathrm{Aut}(X)$ which has property (*) for every partition $\mathscr{L}$. Also it is easy to see that G_γ the stabilizer of a doubly infinite geodesic γ, G_ω the stabilizer of an end $\omega \in \Omega$, or B_ω are groups of this type. In fact $B_\omega = \mathrm{Aut}_{\mathscr{L}}$ where $\mathscr{L}$ is the partition $\{H_n\}$, $n \in \mathbf{Z}$ into the horicycles associated with ω. Therefore every maximal amenable subgroup of $\mathrm{Aut}(X)$ is of type $\mathrm{Aut}_{\mathscr{L}}$ [3, th. 1]. More generally, if G is a subgroup of $\mathrm{Aut}(X)$ and $\mathscr{L}$ is the partition into the orbits of G on X (or on Ω), then $G \subset \mathrm{Aut}_{\mathscr{L}}$ and $\mathrm{Aut}_{\mathscr{L}}$ is the maximal subgroup of $\mathrm{Aut}(X)$ which has the same orbits of G.

Propositions 1 and 2 imply that the dual of B_ω is countable, it has only one accumulation point (the one-dimensional trivial representation) and, except for the trivial representation, every unitary irreducible representation has a dense subspace of coefficients with compact support. More precisely, if $r+1>3$ then there exist irreducible unitary cuspidal representations of B_ω for every Δ with $\operatorname{diam}\Delta \geqq 2$; in fact this is a consequence of Theorem 3 for $\operatorname{diam}\Delta > 2$. For $\operatorname{diam}\Delta = 2$, we can prove directly that $(\tilde{B}_\omega(\Delta))_0^\wedge \neq \emptyset$ because $\tilde{B}_\omega(\Delta)/B_\omega(\Delta)$ is isomorphic to S_r, the group of all permutations of the set $\{1,2,\ldots,r\}$.

If $r+1=3$ and $\operatorname{diam}\Delta=2$, obviously $(\tilde{B}_\omega(\Delta))_0^\wedge=\emptyset$; in fact $\Delta=\{y\in X: d(x_0,y)\leqq 1\}$ for some $x_0\in X$. Let $\{x_0,a_1,a_2,\ldots,a_n,\ldots\}$ be the geodesic joining x_0 to ω. Then $B_\omega(\Delta)=B_\omega(\Delta_i)$ for the two maximal proper subtrees Δ_i different from the edge $[x_0,a_1]$. If $r+1=3$ and $\operatorname{diam}\Delta>2$, then Theorem 3 implies that $(\tilde{B}_\omega(\Delta))_0^\wedge \neq \emptyset$ if and only if the vertex of Δ at "minimal" distance from ω is at distance greater than one from $\partial\Delta^0$ where $\partial\Delta^0$ is the set of vertices of homogeneity one of Δ^0 (recall that Δ^0 is the subtree of Δ of vertices of homogeneity $r+1$ in Δ). In particular, for $r+1=3$, there exist infinitely many Δ such that $(\tilde{B}_\omega(\Delta))_0^\wedge\neq\emptyset$ and infinitely many Δ such that $(\tilde{B}_\omega(\Delta))_0^\wedge=\emptyset$.

The group B_ω is similar to the group of all automorphisms of the following quotient tree X_0 of X: let $\{H_n\}$ be the partition into the horicycles associated with ω, let H be the subgroup which fixes the set $\bigcup_{n<0}H_n$. H is a closed normal subgroup of B_ω and B_ω/H is isomorphic to the group of all isometries of the subtree $X_0=\bigcup_{n\geqq 0}H_n$. The boundary of X_0 consists only of ω, and the dual of $\operatorname{Aut}(X_0)$ is similar to the dual of B_ω. It is enough to consider the geodesic $[x_0,\omega)=\{x_0,x_1,\ldots,x_n,\ldots\}$ (x_0 is a terminal vertex of X_0); let K_n be the stabilizer in $\operatorname{Aut}(X_0)$ of x_n, $n\geqq 0$. Then $\operatorname{Aut}(X_0)=\bigcup_{n=0}^{+\infty}K_n$. In this case it is enough to consider the Gelfand pair $(\operatorname{Aut}(X_0),K_0)$. The spherical functions are the following: ϕ_∞ the function identically one and, for every $n\geqq 0$, the functions $\phi_n=\chi_{K_n}+(1-r)^{-1}\chi_{(K_{n+1}\setminus K_n)}$. The description of the cuspidal representations is similar.

If F is a local field, then the group $\mathrm{PGL}(2,F)$ can be embedded into the group of all isometries of some homogeneous tree [5]. The group G of matrices of the form $\left(\begin{smallmatrix} a & b\\ 0 & c\end{smallmatrix}\right)$ with $|a|=|b|=1$ is contained in B_ω for some $\omega\in\Omega$. For a p-adic field, this group, introduced by Fell, has been studied by G. Mauceri in [2]; it is a semidirect product. The restrictions of the spherical functions of B_ω to G are, exactly, the spherical functions of G. But the cuspidal irreducible representations of G have no coefficients with compact support; they are in fact characters (B_ω has only the trivial character). The group of Fell is more difficult to study and the classification of its irreducible representations relies upon the representation theory of group extensions due to Mackey.

Finally we observe that the isometry k occurring in property $(*)$ is not a matrix because k fixes infinitely many points of Ω. In fact the tree boundary can be regarded as the projective line of F and the action of PGL(2, F) on Ω is the natural action of PGL(2, F) on the projective line. This means that a matrix of PGL(2, F) which fixes three points of Ω is the identity. In particular PGL(2, F) (and so G) does not satisfy the property $(*)$.

REFERENCES

1. S. Lang, $SL_2(\mathbf{R})$, Addison-Wesley, Reading Mass., 1975.

2. G. Mauceri, *Square integrable representations and the Fourier algebra of a unimodular group*, Pacific J. Math. **73** (1977), 143–154.

3. C. Nebbia, *Amenability and Kunze–Stein property for groups acting on a tree*, Pacific J. Math. **135** (1988), 371–380.

4. G. I. Ol'shianskii, *Classification of irreducible representations of groups of automorphisms of Bruhat–Tits trees*, Funktsional'nyi Analiz i Ego Priolozheniya **11** (1) (1977), 32–42.

5. J. P. Serre, *Arbres amalgames* SL_2, Astérisque **46** (1977).

6. J. Tits, *Sur le groupe des automorphismes d'un arbre*, Essays on topology and related topics, Mémoires dédiés a G. de Rham, Springer-Verlag, Berlin, 1970, pp. 188–211.